Critical sets of nonlinear Sturm-Liouville operators of Ambrosetti-Prodi type

H. Bueno and C. Tomei*

Dep. de Matemática - ICEx, UFMG, Caixa Postal 702,
Belo Horizonte, MG 30.123-970, Brazil.
Dep. de Matemática, PUC-Rio, R. Marquês de S. Vicente 225
Rio de Janeiro, RJ 22453-900, Brazil.
email - H. Bueno: hamilton@mat.ufmg.br C. Tomei: tomei@mat.puc-rio.br

Mathematics Subject Classification: 34A26, 34B24, 34L30.
Keywords: Nonlinear Sturm-Liouville operator, critical set, convex nonlinearities.

Abstract

The critical set C of the operator $F : H_D^2([0,\pi]) \to L^2([0,\pi])$ defined by $F(u) = -u'' + f(u)$ is studied. Here $X := H_D^2([0,\pi])$ stands for the set of functions that satisfy the Dirichlet boundary conditions and whose derivatives are in $L^2([0,\pi])$. For generic nonlinearities f, $C = \cup C_k$ decomposes into manifolds of codimension 1 in X. If $f'' < 0$ or $f'' > 0$, the set C_j is shown to be non-empty if, and only if, $-j^2$ (the j-th eigenvalue of u'') is in the range of f'. The critical components C_k are (topological) hyperplanes.

Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain, $f \in C^k(\mathbb{R})$, $k \geq 2$, and $g : \Omega \to \mathbb{R}$ be two given functions. Since the pioneering work of Ambrosetti-Prodi ([1]), the nonlinear problem

$$-\Delta u + f(u) = g, \quad u|_{\partial\Omega} = 0,$$

has been studied by various methods of nonlinear analysis in the case

$$\sigma(-\Delta) \cap f'(\mathbb{R}) \neq \emptyset.$$

($\sigma(-\Delta)$ stands for the spectrum of $-\Delta$).

In this paper we deal with the one dimensional version of this problem, namely

$$-u'' + f(u) = g(t), \quad u(0) = u(\pi) = 0. \tag{1}$$

There exists an extensive literature concerning this problem (see, for example, [4], [8], and references therein). Usually one obtains *a priori* estimates on the nonlinearity, which result in bounds on the number of solutions of (1).

Despite the success of the different methods used in this problem, they do not give insight into the nature of the change in the number of solutions of

*Supported in part by CNPq-Brazil, FINEP and FAPERJ.

(1), which was one of the major features of the work of Ambrosetti and Prodi, as well as subsequent work of Berger and Podolak ([2]). There is a certain reappraisal in recent years ([3], [7],[8]) of the Ambrosetti-Prodi method. This paper, whose approach is also geometric, may be seen as a portion of a larger project, inspired by some techniques and methods also present in [6]. For a first order differential equation, these authors characterized the critical set C, then studied of the stratification of C by different Morin singularities and finally considered the geometry of the image of the critical set.

Here we address only the first part of this project: for convex or concave nonlinearities, we characterize the critical set C of the Sturm-Liouville operator

$$\begin{array}{rccc} F: & X & \to & Y \\ & u & \mapsto & -u'' + f(u), \end{array}$$

defined in the Sobolev space $X := H^2_D([0, \pi])$ of functions that satisfy the Dirichlet boundary conditions and whose second derivatives are in $Y := L^2([0, \pi])$. The results also hold for different pairs of spaces without difficulty.

Our main result may be synthesized as follows

Theorem A: *Suppose that the nonlinearity f satisfies $f'' > 0$ or $f'' < 0$. Then the critical set C of the operator F decomposes into connected components C_j, associated to the free eigenvalues $\{1^2, 2^2, \dots\}$ belonging to the range of $-f'$. Each C_k is a (topological) hyperplane, which admits a simple explicit parametrization by functions of average zero.*

If 1 is the only square integer in the range of $-f'$, Theorem A is part of the proof of the original Ambrosetti-Prodi Theorem. If f' crosses only the first m square integers, the result was obtained by Ruf ([8],Proposition 9), by using different techniques. In a forthcoming paper, Burghelea, Saldanha and Tomei prove that, for arbitrary generic nonlinearities, the components of the critical set are still topological hyperplanes parametrized by square integers in the range of $-f'$. The arguments do not have the same expliciteness than those presented in this paper, and depend strongly on special properties of infinite dimensional topology.

In a sense, this paper is a nonlinear version of oscillation theory: as the function u varies in X, consider the argument of a nonzero solution of the linearized equation $-v'' + f'(u)v = 0,\ v(0) = 0$ at $t = \pi$. It turns out that this argument has monotonicity properties similar to those of the usual argument of a Sturm-Liouville solution when the potential varies so that it is increased pointwise.

We are not concerned, in this paper, with the study of the image of the critical sets C_k. In the n-dimensional case, for $f'' > 0$ or $f'' < 0$, the image of $F(C_1)$ is studied in [3]. For arbitrary interactions between f' and $\sigma(\Delta)$, the image $N := F(C_1)$ turns out to be a codimension 1 manifold, which is globally parametrized by the functions of average zero. For functions g on one side of N there is no solution $u \in X$ for the equation $F(u) = g$, while on the other side there is at least one solution. This result was obtained by Berger-Podolak ([2]) in the original Ambrosetti-Prodi context, i.e., when $C = C_1$.

When f' interacts with the j first eigenvalues of the free Laplacian, the classification of the singularities in C_k is still an open problem, even in the one

dimensional case. It is well known that C_1 consists only of fold points. Also, higher singularities do appear in C_k, $k = 2, \dots, j$, but the Morin type of the singularities which may occur is unknown (see [8]).

1 Statements and proofs

A simple computation obtains the derivative of F at u,

$$\begin{array}{rccc} DF(u): & X & \to & Y \\ & w & \mapsto & -w'' + f'(u)w. \end{array}$$

The characterization of the critical points of F is then a consequence of Fredholm theory applied to Sturm-Liouville operators. Thus, $u \in C$ if, and only if, the kernel of $DF(u)$ is non-trivial. For Dirichlet boundary conditions, the spectrum is simple. Define $v(u)(t)$ as the solution of the linearized equation

$$-[v(u)]''(t) + f'(u(t))v(u)(t) = 0, \quad v(u)(0) = 0, \ [v(u)]'(0) = 1. \tag{2}$$

The simplicity of the spectrum of $DF(u)$ guarantees that, if $v(u)(\pi) = 0$, then $\ker DF(u)$ is spanned by $v(u)(t)$. Following Prüfer ([5]), let $W : X \times [0, \pi] \to \mathbb{R}$ be the continuously defined argument of the planar vector $([v(u)]'(t), v(u)(t))$, with $W(u)(0) = 0$. It follows that $u \in C$, the critical set, if, and only if, $W(u)(\pi) = k\pi$, $k \in \mathbb{N}^* := \{1, 2, 3, \dots\}$. We define, for all $\theta \in \mathbb{R}$, the level sets

$$M_\theta := \{u \in X;\, W(u)(\pi) = \theta\}.$$

The critical set then decomposes into

$$C_k := M_{k\pi} = \{u \in X;\, W(u)(\pi) = k\pi, \ k \in \mathbb{N}^*\}.$$

Theorem B: *Let $f \in C^r(\mathbb{R})$, $r \geq 2$ be a function such that $f''(0) \neq 0$. Then M_θ (in particular, each C_k) is either empty or a C^r-manifold of codimension 1 in X. If, however, $f''(0) = 0$ and the two conditions below hold,*

(a) the root 0 of f'' is isolated,

(b) $f'(0) \neq -j^2$, $j \in \mathbb{N}^*$,

then the same conclusion is valid for the critical sets C_k.

Proof. We calculate $DW(u)(\pi)$. Differentiation produces

$$DW(u)(\pi) \cdot \varphi = \frac{[v(u)]'(\pi).[Dv(u) \cdot \varphi](\pi) - v(u)(\pi).[Dv(u) \cdot \varphi]'(\pi)}{\{v(u)(\pi)\}^2 + \{[v(u)]'(\pi)\}^2}.$$

In order to obtain $[Dv(u) \cdot \varphi](\pi)$ and $[Dv(u) \cdot \varphi]'(\pi)$, we differentiate problem (2) with respect to u: for all $\varphi \in X^2_D$,

$$\begin{array}{l} -D\big([v(u)]''\big) \cdot \varphi + [f''(u).\varphi].v(u) + f'(u).(Dv(u) \cdot \varphi) = 0 \\ (Dv(u) \cdot \varphi)(0) = 0, \quad [Dv(u) \cdot \varphi]'(0) = 0, \end{array}$$

Denoting $\mu = \mu(u, \varphi) = Dv(u) \cdot \varphi \in X^2$, we are thus led to the initial value problem

$$\begin{aligned} &-\mu'' + f'(u).\mu = -[f''(u).\varphi].v(u) \\ &\mu(0) = 0, \quad \mu'(0) = 0, \end{aligned}$$

which can be solved by variation of constants. The expressions we want to calculate are evaluations of μ and μ' at π. We find

$$DW(u)(\pi) \cdot \varphi = \frac{-1}{\{v(u)(\pi)\}^2 + \{[v(u)]'(\pi)\}^2} \int_0^\pi f''(u(r)).\varphi(r).[v(u)(r)]^2 \ dr.$$

Thus $DW(u)(\pi) \equiv 0$ if, and only if, $f''(u) \equiv 0$, which may happen only if $f''(0) = 0$. So, if $f''(0) \neq 0$, there exists $\varphi \in X_D^2$ such that $DW(u)(\pi) \cdot \varphi \neq 0$ and the result now follows from the Implicit Function Theorem.

Suppose now $f''(0) = 0$. Since 0 is an isolated root of f'' and we must have $f''(u(t)) \equiv 0$, it follows that $u \equiv 0$. But, in this case, denoting $c = f'(0)$, we see that $v = v(u)$ solves

$$-v'' + cv = 0, \quad v(0) = 0, \quad v'(0) = 1$$

and $v(u)(\pi) \neq 0$ if and only if $c \neq -j^2$. Again, C_k is a manifold. $\square$

Let us now consider the Ambrosetti-Prodi context $f'' > 0$ or $f'' < 0$. Let $p(t) \in X$ be any strictly positive function in $(0, \pi)$ and V be the space spanned by $p(t)$. Decompose $X = H \oplus V$ in L^2-orthogonal terms.

Theorem C: *If $f'' > 0$ or $f'' < 0$, then, for $j \in \mathbb{N}^*$, $C_j \neq \emptyset$ if and only if $-j^2$ belongs to the interior of the image of f'. Also, if $M_\theta \neq \emptyset$ (in particular, $C_k = M_{k\pi} \neq \emptyset$), then the projection*

$$\Pi : X = H \oplus V \to H.$$

is a diffeomorphism from M_θ to H.

Proof. For each $\lambda \in \mathbb{R}$ and $h \in H$ fixed, we consider the straight line $\ell_{h,p} = h + \lambda p \in X$. We will show that $\ell_{h,p}$ always intercepts each manifold M_θ once and transversally. Uniqueness and transversality follows from

$$DW(u)(\pi) \cdot p = \frac{-1}{\{v(u)(\pi)\}^2 + \{[v(u)]'(\pi)\}^2} \int_0^\pi f''(u(r))p(r)[v(u)(r)]^2 \ dr \neq 0.$$

Smoothness (and local smooth invertibility, for a fixed θ) in h, in turn, follows by setting $u = h + \lambda p$ in the formula above.

Suppose that $f'' < 0$: in this case, $-f'$ is increasing, $-f'(-\infty) = a$ and $-f'(\infty) = b$. Here $a, b \in [-\infty, \infty]$. Clearly, the range (a, b) of $-f'$ is an open set. From standard oscillation theory, the solutions of the three problems below (with initial position and velocity at 0 equal respectively to 0 and 1) have increasing arguments,

$$v_a'' + av_a = 0 \quad v_\lambda'' - f'(h + \lambda p)v_\lambda = 0 \quad v_b'' + bv_b = 0.$$

We will see that the asymptotics of the argument at both ends of $\ell_{h,p}$ will be given by the argument of the solutions of the leftmost and rightmost problems.

For a fixed value of the parameter ω, let v_ω be the solution of the problem

$$v'' + \omega v = 0, \quad v(0) = 0,\ v'(0) = 1.$$

We denote

$$W_1 = W_1(\omega, t) = W(v_\omega)(t) \quad \text{and} \quad W_2 = W_2(\lambda, t) = W(h + \lambda p)(t).$$

It follows immediately that

$$\lim_{\omega \to -\infty} W_1(\omega, \pi) = 0 \quad \text{and} \quad \lim_{\omega \to \infty} W_1(\omega, \pi) = \infty.$$

We first study the behavior of $W_2(\lambda, t)$ when $\lambda \to \infty$. Fix $\omega \in \mathbb{R}$. It is easy to prove (see [5]) that the argument function satisfies the differential equation

$$[W(u)(t)]' = \cos^2 W(u)(t) - f'(u(t)) \sin^2 W(u)(t).$$

Then

$$(W_1 - W_2)' = [\omega + f'(h + \lambda p)] \sin^2 W_2 + \left(\frac{\cos^2 W_1 - \cos^2 W_2}{W_1 - W_2} \right) (W_1 - W_2).$$

Thus $U' + gU = H$, for $U := W_1 - W_2$, $H =: [\omega + f'(h + \lambda p))] \sin^2 W_2$ and

$$g := -\left(\frac{\cos^2 W_1 - \cos^2 W_2}{W_1 - W_2} \right).$$

Since g is continuous and uniformly bounded in λ, we can define the integrating factor $G(t) = \exp(\int_0^t g(s) ds)$. Multiplication by this factor and integration produces

$$\exp(G(\pi)) U(\pi) = \int_0^\pi \exp(G(t)) H(t) dt. \tag{3}$$

If $b < \infty$, we choose $\omega = b$ and have $H(t) > 0$, yielding $W_1(b) > W_2(\lambda)$ for all λ. The estimate

$$\int_0^\pi \exp(G(t)) H(t) dt \le \left(M \| \sin^2 W_2 \|_{L^2} \right) \| b - (-f'(h + \lambda p)) \|_{L^2},$$

where $M := \max_{t \in [0,\pi]} \exp(G(t))$, shows that $U(\pi) \to 0$ when $\lambda \to \infty$. This shows that $W_2(\lambda, \pi)$ converges increasingly to $W_1(b, \pi)$, for $\lambda \to \infty$, if $b < \infty$. If $b = \infty$, we have $-f'(h + \lambda p) - \omega > 0$ in the interval $(\delta, \pi - \delta)$ if λ is big enough. Defining $r(t) := \sin^2 W_2 \exp(G(t))$, we obtain

$$\exp(G(\pi))(U(\pi)) \le 2M\delta\omega - \int_\delta^{\pi - \delta} [-f'(h + \lambda p) - \omega] r(t) dt < 0$$

when $\lambda \to \infty$. Consequently, the integral in (3) is negative, if λ is sufficiently big. We conclude that $W_2(\lambda, \pi) > W_1(\omega, \pi)$. Since $\lim_{\omega \to \infty} W_1(\omega, \pi) = \infty$, and we again have that $W_2(\lambda, \pi)$ increases to $W_1(b, \pi) = \infty$.

In a similar fashion, we obtain that $W_2(\lambda, \pi)$ converges decreasingly to $W_1(a, \pi)$, when $\lambda \to -\infty$. Thus, the line $\ell_{h,p}$ intersects the same manifolds M_θ irrespective of h: the values of θ for which $\ell h, p$ trespasses M_θ lie strictly between a and b. The case $f'' > 0$ is similar. $\square$

References

[1] A. Ambrosetti and G. Prodi: *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura Appl. **93** (1972), 231-246.

[2] M. S. Berger and E. Podolak: *On the solutions of a nonlinear Dirichlet problem*, Indiana Univ. Math. J. **24** (1975), 837-846.

[3] H. Bueno and A. Zumpano: *Remarks about the geometry of the Ambrosetti-Prodi problem*, to appear in Nonlinear Analysis, TMA.

[4] V. Cafagna: *Global invertibility and finite solvability*, Nonlinear Funct. Anal. Lecture Notes Pure Appl. Math. Vol. 121, Ed. P. S. Milojević, Marcel Dekker, New York, 1990.

[5] E. Coddington and N. Levinson: Theory of Ordinary Differential Equations, McGraw-Hill Book Company (1955), T M H Edition, Bombay.

[6] I. Malta, N. C. Saldanha and C. Tomei: *Morin singularities and global geometry in a class of ordinary differential operators*, Top. Meth. Nonlinear Anal. **10** (1997), 137-169.

[7] H. McKean: *Singularities of a simple elliptic operator*, J. Differential Geometry **25** (1987), 157-165 and *Correction to "Singularities of a simple elliptic operator"*, J. Differential Geometry **36** (1992), 255.

[8] B. Ruf: *Singularity theory and bifurcation phenomena in differential equations*, Topological nonlinear Analysis II (Frascati, 1995), 315-395, Progr. Nonlinear Differential Equations Appl. **27**, Birkhäuser Boston, Boston, 1997.